Superfluid Helium, Bose-Einstein Statistics, and Bose-Einstein Condensation

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The discover of superfluid helium in 1938 brought with it a revolutionary need to understand the macro and micoscopic qualities of this novel fluid system. In understanding the superfluid phase of ^{4}He, we relate superfluid phase transition to Bose statistics and Bose-Einstein Condensation. In this work we cover the unique nature of superfluid helium along with its macroscropic properties, we formulate a strict mathematical connection between Bose statistics/condensation and the quantum understanding of superfluids, and we synthesize the modern understanding of the relationship between the two phenomena and point to future research areas and possible breakthroughs.

I. Introduction and Motivation

One of the most 20th century's most astounding physical breakthroughs was the experimental discovery of superfluidity in 1938 by Kapitza [1]. In this year, superfluid ^{4}He (known as Helium-II in its superfluid form) was independently realized in two different labs on the European continent. This new liquid state was the result cooling the liquid ^{4}He below its lamba point of 2.18 K, after which, new macroscropic quantum properties of the liquid, ie. the sudden drop of its viscosity to zero, are able to be seen by the naked eye. Before describing in greater detail the descriptions of the liquid that followed, it its worth situating this discovery in its respective time with the state-of-the art in statistical physics and condensed matter theory.

The most relevant theoretical discovery to superfluidity, was the new phase of matter proposed by Einstein in 1925, Bose-Einstein condensation (BEC). Einstein theorized the existence of this state of matter after his generalization of Satyendra Bose's statistical description of photonic (generalized by Einstein to include bosonic particles) state distributions. The BEC represented a special case of Bose-Einstein statistics in which at ultra low temperatures below a critical point, the number of particles occupying the lowest state of the system becomes so large that wavefunction interference and other quantum phenomena lead to a new microscopically distinct phase of matter. In other words, the distance between particle spacing in the condensate becomes much smaller than the quantum volume and therefore quantum statistics play an important part in determining the state's properties. On an intuitive level, the fact that the critical point of superfluid Helium and the theorized point for a BEC were almost identical, ^{4}He as a boson obeyed bosonic statistics, the similar overloading of low energy quantum occupation (and the strange phenomena that follow), and that the discoveries of the two states of matter were pioneered in very quick temporal succession, leads one to ask how BEC and superfluidity are related. As mentioned, the connection seemed relatively obvious, yet as BEC were not experimentally realized until 1995, much of the theory of superfluidity in ^{4}He, and other materials, by Landau, Bogoluibov, and others did not even touch on bosonic statistics or condensation. This discrepancy in the theories of superfluidity and BEC leaves much unanswered related to the fundamental physics of these novel phases of matter. The solutions to these questions of unification and investigation of such topics are extremely important, with applications in astrophysics, quantum gravity, condensed matter, and beyond. This paper aims to highlight the relationship between bosonic statistics and superfluidity from their discoveries to their modern day understanding.

This report on superfluid Helium, Bose statistics, and BEC's will be structured as follows. Section II provides a description of the macroscopic properties of superfluid ^{4}He while delving into some of the paradoxical quantum behavior of the substance. Section III is devoted to describing the details of bosonic statistics along with the macro/microscopic properties of BEC's. Section IV elaborates upon the theoretical connection between superfluidity, bosonic statistics, and BEC's. Section V summarizes the report: detailing its conclusions and providing outlook and discussion based on its investigation.

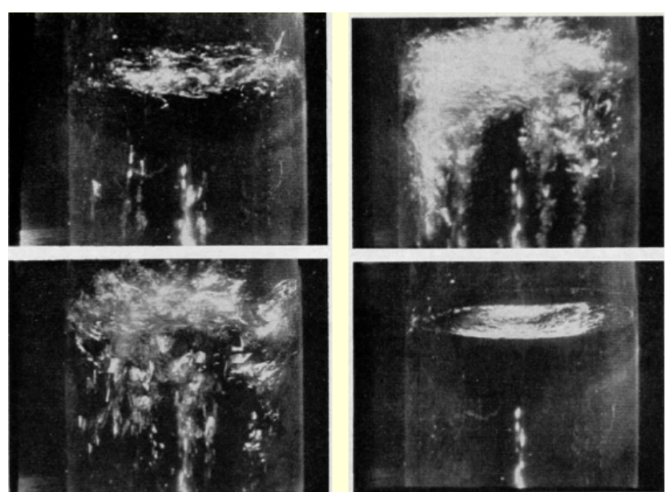

FIG. 1. These pictures show the Helium as one goes from above the superfluid transition temperature(lambda point) to below. The dramatic cessation of boiling is a consequence of hydrodynamic change in the occupancy of the ground state setting in (from [2]).

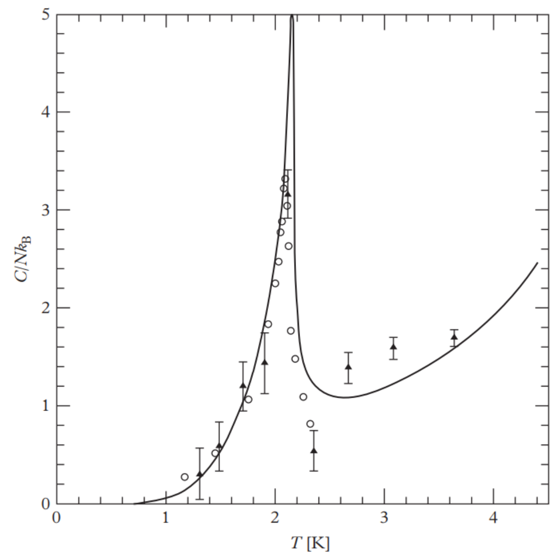

FIG. 2. Specific heat of ^{4}He. Solid line—experiment at saturated vapour pressure; triangles with error bars—PIMC calculations. From Ceperley (1995).

II. Superfluidity and ^{4}H

To summarize the exotic nature of superfluid helium-II, we will begin by elucidating on the transition (or lambda point) mentioned above, at which ^{4}He transitions from its super-cooled Helium-I state beyond the lamba point into its superfluid helium-II state. It must first be noted, that one of the reasons Helium was the first liquid to realize a superfluid state, and that has made it the first and ideal candidate to study this state of matter, is because it is the only permanent liquid in nature. In other words, it is the only element such that when its liquid form is cooled close to absolute zero at the atmospheric pressure of earth it will remain a liquid. The helium-I to helium-II transition process is seen clearly in Figure 1 where the rapid boiling of the supercooled liquid abruptly calms to standstill as the macroscopic properties of the system experiences a breaking of gauge symmetry. Elaborating on the lambda point where this transition occurs, Figure 2 clearly shows the origin of this name as the graph of the specific heat over temperature shows a divergence of a lambda shape at Tc. This divergence at the transition point marks a second order phase transition (a similar type of transition is seen in BEC, ferromagnetic, and superconducting transitions)[2], in which an order parameter describing the superfluid properties of the system goes discontinuously from zero to some value as the substance passes below the critical temperature. After passing the lambda point, in many ways the ^{4}He is an entirely new liquid in the properties it displays. Helium-II has the property of irrationality, meaning the $\nabla \times \vec{v} = 0$ everywhere with $\vec{v}$ defined as the superfluid velocity. This is more nuanced than it may originally seem as above absolute zero not all of the fluid is a superfluid, however, irrotationality for the bulk of the liquid holds. Helium II also is a zero entropy liquid as it displays no temperature gradient corresponding to its heat transfer with other matter. The fascinating fact the superfluid has zero entropy and temperature gradient combines with other fascinating macroscopic properties of the state in the fountain effect (Figure 3). Another fascinating result of the zero temperature gradient and superfluid nature is that Helium-II has the highest thermal conductivity of any material (100 times greater than copper). The fluid also displays special capillary flow as a result of the lack of diffusion due to the zero viscosity of the liquid: allowing the fluid to pass through previous stoppages as well climb up the fluid's container walls. The film over the walls is specifically known as Rollin's film (Fig. 4). All of these characteristics are results of the second order phase transition at the lambda point and the novel excitation and fluctuations the superfluid displays in its unified state. There is much mathematical detail that explains these characteristics and their fluctuations, most of it described by second quantization and quantum field operators outside of the scope of this paper, yet some of this detail will be worked out in section IV.

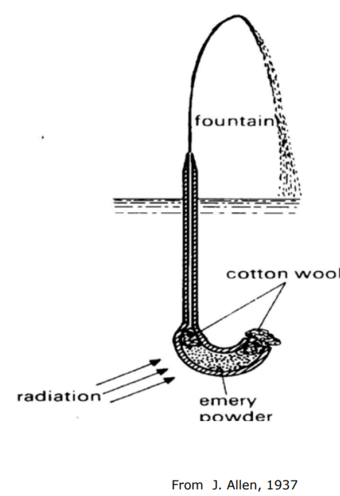

FIG. 3. In the fountain effect, an empty capsule is placed with its top above the surface layer of the superfluid. The bottom of the capsule is stuffed with cotton and emery powder such that before the liquid is in superfluid phase the bottom is impermeable. However, when the capsule is placed in superfluid ^{4}Hea and is struck with radiation heating it up, the superfluid flows upward against gravity through the capillaries due to the capillary and viscous forces. The superfluid continues to flow through the bottom of the capsule until the top is filled and begins to overflow in a fountain. This effect continues as long as the heating of the capsule is maintained and the temperature of the superfluid mass remains below the lambda point.

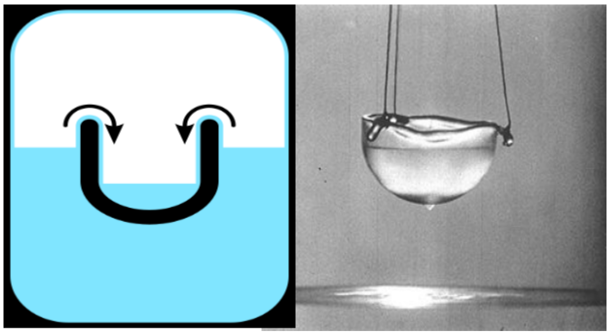

FIG. 4. Left: cartoon demonstration superfluid Rollin's film development and motion as a means of equalizing the surface level. Right: experimental photo showing the Rollins film dripping down the outside of a container into the greater superfluid reservoir below.

III. Bose-Einstein Statistics and Condensates

To develop our understanding of superfluidity we highlight some important properties of Bose statistics and BEC's. Beginning with the famous Bose-Einstein Distribution for an ideal Bose gas, we write the average boson occupancy number for a state with with energy ϵ_i interacting with a reservoir with which the system is in thermal equilibrium as:

$$\overline{n}_{\epsilon_i} = \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1}, \tag{1}$$

where $\beta = 1/k_b T$ and μ is the chemical potential of the reservoir. The total number of particles follows easily.

$$N = \sum_i \overline{n}_{\epsilon_i}. \tag{2}$$

The average occupancy equation provides and important physical constraint $\mu < \epsilon_0$ on the chemical potential of the ideal Bose gas reservoir and the lowest possible energy state of the single particles. The violation of this constraint would result in a negative occupancy number for all states with energy less than μ. As $\mu \rightarrow \epsilon_0$ the occupation number for the lowest energy state

$$N_0 \equiv \overline{n}_{\epsilon_0} = \frac{1}{e^{\beta(\epsilon_0 - \mu)} - 1} \tag{3}$$

increases to infinity. This constraint, and the asymptotic approach of $\mu \rightarrow \epsilon_0$ is what is responsible for Bose-Einstein condensation. If we choose to write the total number of particles in the system as

$$N = N_0 + N_T, \tag{4}$$

with

$$N_T(T, \mu) \equiv \sum_{i \neq 0} \overline{n}_{\epsilon_i}(T, \mu) \tag{5}$$

defined as the number of particles in all of the states aside from the ground state. For a fixed value of T, N_T approaches its maximum value at $\mu = \epsilon_0$, however as we said above, the value of N_o tends to infinity. With T varying, it follows that the massive increase in contribution to the wavefunction from the condensate occurs only when T moves lower than the critical temperature T_c. Prior to passing the critical temperature, the condensate fraction N_0/N is zero, however once past this point, the ratio increases until it reaches a theoretical maximum of 1 at T = 0 K. A strange phenomena related to the superfluid lambda point mentioned above is that the specific heat of the ideal Bose gas goes to 0 after the temperature passes the critical point. The ideal gas solution also suggests infinite compressibility once the gas becomes a condensate, however solutions incorporating two-body interaction and dilution in the Bose gas give more realistic expectations using interatomic forces in the quantum limit. The quantum treatment of the BEC also leads to complex vortex and soliton solutions at T $\neq$ 0 K which are present in superfluid theory as well [3–5].

IV. Superfluidity

Before defining the details of superfluid theory and its pioneering by Lev Landau, we must introduce the order parameter mentioned in Section II. We do this using second quantization to write the field operator $\hat{\Psi}(\mathbf{r})$, with the standard single-particle wave functions $\psi_i(\mathbf{r})$, in the form

$$\hat{\Psi}(\mathbf{r}) = \sum_i \psi i(\mathbf{r}) \hat{a}_i \tag{6}$$

where $\hat{a}_i$ ($\hat{a}_i^\dagger$) are the annihilation (creation) operators of a particle in particlar state ψ_i and obey the commutation relations (analogous to simple harmonic oscillator)

$$[\hat{a}_i, \hat{a}_i^\dagger] = \delta_{ij}, [\hat{a}_i, \hat{a}_j] = 0, [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0. \tag{7}$$

These operators are normalized such that $\langle \hat{a}_j^\dagger \hat{a}_i \rangle = \delta_{ij} N$ is the number operator. This field operator can be loosely understood as the sum of all possible ways of annihilating one particle at $\mathbf{r}$ through any basis states of the $\psi_i(\mathbf{r})$ wave functions. For understanding BEC and superfluidity, it is helpful to separate terms of the sum so that the $i = 0$ term corresponding to the lowest (condensate) state is distinct.

$$\hat{\Psi}(\mathbf{r}) = \psi_0(\mathbf{r}) \hat{a_0} + \sum_{i \neq 0} \psi_i(\mathbf{r}) \hat{a}_i \tag{8}$$

We then act on this equation using the Bogoliubov approximation [3, 6], in which the operators $\hat{a}_i$ *and* $\hat{a}_i^\dagger$ are replaced with the complex number $\sqrt{N_0}$ such that the field operator becomes

$$\hat{\Psi}(\mathbf{r}) = \sqrt{N_0} \psi_0(\mathbf{r}) + \sum_{i \neq 0} \psi_i(\mathbf{r}) \hat{a}_i. \tag{9}$$

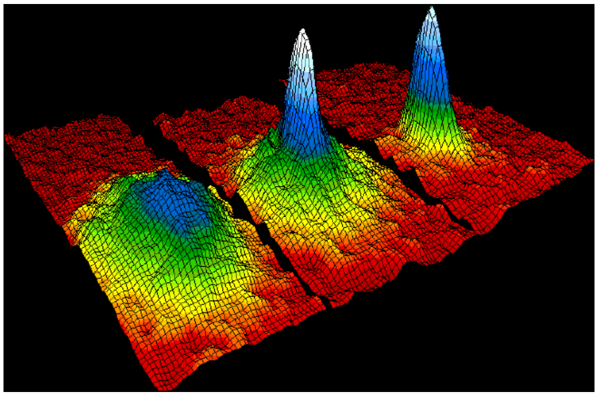

FIG. 5. Distribution of velocities for a gas of rubidium atoms. Left: prior to condensation (T $> T_c$). Center: Just after the appearance of the condensate. Right: Almost entirely pure condensate with velocities unified.

This approximation is appropriate in its use to describe the macroscopic properties of a majority condensate system (superfluid ^{4}He, BEC's) where $N_0 = \langle \hat{a}_j^\dagger \hat{a}_i \rangle >> 1$. Using this approximation and ignoring the non-condensate components of the field operator (all i > 0), the wave function of the condensate may be defined as $\hat{\Psi}_0(\mathbf{r})$, such that

$$\Psi_0(\mathbf{r}) = \sqrt{N_0}\psi_0(\mathbf{r}) = |\Psi_0(\mathbf{r})|e^{iS(\mathbf{r})}. \qquad (10)$$

The added phase component in the last term reflects the 1-D gauge symmetry the wave function of the condensate (AKA the order parameter) naturally possesses. The phase $S(\mathbf{r})$ is extremely important in condensate and superfluid phenomena. Making an explicit choice for the phase of the order parameter in its $\mathbf{r}$ dependant phase corresponds to the breaking of this gauge symmetry. The Boguliubov approximation for the field operator is such that the expectation value $\langle N| \hat{\Psi} |N\rangle \neq 0$. Using the normal annihilation rules for the expectation value of (6), the expectation value would always be assumed to be zero due to the orthogonality of the $\langle N\rangle\, N-1$ states (for standard size N). However, the Bogoliubov approximation removes the annihilation operator from the condensate wave function and instead simply keeps the normalization term $\sqrt{N_0}$ the $\hat{a_0}$ would have pulled out when acting on $|N_0\rangle$. The physical interpretation of this approximation is that as $N_0 >> 1$ it plays the role of a reservoir as the annihilation of a single particle makes no change in the properties of the system. The time dependence of the order parameter is independent of the energy of the ground state and can be shown to be

$$\Psi_0(\mathbf{r}, t) = \Psi_0(\mathbf{r})e^{i\mu t/\hbar}, \qquad (11)$$

where the chemical potential $\mu = E(N) - E(N-1)$ [5, 6].

Much of Landau's original superfluid theory work is based on the effects of Galilean transformations [7, 8]. To demonstrate a mathematical connection between superfluidity and BEC's, it is useful to consider the condensate wave function (order parameter) under Galilean transformation. Even though the condensate system is uniform and the density of the condensate function $|\Psi_0|^2$ is constant, the wavefunction itself varies under such a transformation due to its locally dependant phase factor. We can incorporate this into the phase by defining the Galilean transform of the condensate wave function using the Heisenberg representation and the fact that[6] pg. 69). We define the transformation $\Psi_0'(\mathbf{r}, t)$ in the form

$$\Psi_0'(\mathbf{r}, t) = \Psi_0'(\mathbf{r} - \mathbf{v}t, t)\exp[\frac{i}{\hbar}(m\mathbf{v}\cdot\mathbf{r} - \frac{1}{2}mv^2 t)], \qquad (12)$$

with the constant velocity vector$\mathbf{v}$ as a solution. Examining our order parameter we remember that with in a stable coordinate system with no r dependent phase the condensate wave function of the uniform fluid is $\Psi_0 = \sqrt{N_0}e^{-i\mu t/\hbar}$. However under the Galilean transformation with the constant velocity defined above, the wave function takes the r dependant phase in the form $\Psi_0 = \sqrt{N_0}e^{iS(\mathbf{r})}$, with

$$S(\mathbf{r}, t) = \frac{1}{\hbar}[m\mathbf{v}\cdot\mathbf{r} - (\frac{1}{2}mv^2 + \mu)t]. \qquad (13)$$

Equation (13) shows that the velocity is related to the gradient of the phase by

$$\mathbf{v}_s = \frac{\hbar}{m}\nabla S. \qquad (14)$$

This velocity can be idendified as the superfluid velocity and establishes the irrationality condition ($\nabla \times \nabla S = 0$) that is crucial in the unique characteristics displayed by the superfluid. Another triumph of this result is that in the superfluid velocity derivation, no assumptions about the details of the liquid/gas were made in terms of it dilution or zero temperature constraints. The result is a consequence of BEC condensation and the macroscopic properties displayed by the order parameter. This identification of the superfluid velocity with the gradient of the phase of the order parameter is one of the key characteristics of the relationship between BEC's and superfluids. However this does not entail identity between the two, it must also be noted that there are differences in the theoretical properties of the substances. For example, the theoretic densities of the two states differ as they approach the 0 K limit due to varying quantum effects. Another example of a potential difference between the the two phenomenon, and specifically BEC's and Helium-II, is the use of rotons and second and third sound to describe micro and macroscopic properties of helium-II which do not yet have an analog in BEC's in a gas form [5]. Microscopically, and specifically in the behavior and use of the order parameter to describe the BEC and superfluid second-order phase transitions, much is common. Yet attempts at providing exact description of their sameness and difference is ongoing to this day. Perhaps the discussion is best summarized by Beliav at the end of a landmark paper [9] on interacting Bose-condensed gases: *The difference between liquid helium* *and a non-ideal Bose gas is only a quantitative one, and no qualitatively new phenomenon is expected to arise in the transition from gas to liquid.*

V. Summary and Outlook

The crux of this paper has been trying to make definitive statements about the relationship between bosonic condensation and superfluidity. Both phenomena are the ultimate result of bosonic statistics and the passing of a critical temperature near absolute zero. Both phenomena are also second-order phase transitions with drastic symmetry breaking and novel macroscopic properties emerging from the unified quantum state. And the list of similarities goes on and on, and yet, as is said in [2], Bose liquids easy to measure as superfluids and hard to measure as BEC's, while Bose gases are easy to measure as BEC's and hard to Measure as superfluids. Another theoretical discrepency arises when we note that weakly interacting Bose gases do fufill the Landua criterion of a superfluid, while an ideal bose gas does not. This seems to suggest that a great deal of progress must be made in further aligning the real (not idealized) forms of Bose statistics in BEC's and their quantum interaction to have a more physically concrete and generalizable criterion and mapping between the two. There are direct parallels between the phenomena and if the understanding of the liquid to gas transition was 100 percent possible theoretically or experimentally we could finally close the subject on their identity. Yet for now they remain disjoint, continuing to be studied and related in great detail in their macroscopic quantum phenomena and applications towards new physics. The movement for future research continues to be to work on the theoretical and experimental investigation into the similarities between the advanced field theories and macroscopic characteristics involved in Bose liquid condensates or superfluid gases in the lab. Steps in and outside the lab will need to be made in continuing to find a place for these exotic states of matter in the physical paradigm as their importance and connection to other phenomena from a variety of physical subjects increases.

[1] P. Kapitza, Viscosity of liquid helium below the λ-point, Nature **141**, 74 (1938).

[2] A. Griffin, New light on the intriguing history of superfluidity in liquid4he, Journal of Physics: Condensed Matter **21**, 164220 (2009).

[3] N. Bogoliubov, On the theory of superfluidity, J. Phys **11**, 23 (1947).

[4] A. Griffin, G. Allan, *et al.*, *Excitations in a Bose-condensed Liquid*, Vol. 4 (Cambridge University Press, 1993).

[5] P. Nozieres, *Theory Of Quantum Liquids: Superfluid Bose Liquids* (CRC Press, 2018).

[6] L. Pitaevskii and S. Stringari, *Bose-Einstein condensation and superfluidity*, Vol. 164 (Oxford University Press, 2016).

[7] L. Landau, Theory of the superfluidity of helium ii, Physical Review **60**, 356 (1941).

[8] L. Landau, On the theory of superfluidity, Physical Review **75**, 884 (1949).

[9] S. Beliaev, Energy spectrum of a non-ideal bose gas, Sov. Phys. JETP **34**, 299 (1958).